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A robust method for regenerative heat exchanger calculations

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Abstract—A stable and accurate closed method for the solution of the differential equations describing thermal regenerator behaviour is discussed. The approach adopted is related to that proposed by Razelos [*Wärme- u. Stoffübertr.* **12**, 59–71 (1979)], but employs the trapezoidal rule to provide accuracy and, in particular, numerical stability. Details of the new method are described together with its application to a large range of the descriptive parameters.

1. INTRODUCTION

THE OPERATION of a thermal regenerator can be considered to be the continuous, alternate passage of hot and cold fluid streams over a solid matrix or packing. The length of time for which each fluid flows is known as a period. The packing facilitates heat transfer between the hot and cold fluids by absorbing thermal energy during the hot period and releasing some of this stored energy during the cold period, in order to warm the cold fluid. At the end of each period it is assumed that any remaining fluid in the channels of the matrix is expelled before the start of the next period in what is known as a reversal period. The combination of the hot and cold periods together with the reversal periods forms a cycle. After a sufficient number of such cycles the regenerator reaches a state of dynamic equilibrium, where the chronological variation of the fluid and solid temperatures is identical over successive cycles. The directions of the hot and cold fluid streams defines the mode of operation of a regenerator. If both streams flow in the same direction the mode is co-current or parallel-flow and in opposite directions is counter-current or counter-flow. The latter is more common and is discussed in greater detail in this paper.

In 1979 Razelos [1] presented a closed solution to the two linear, partial differential equations which describe thermal regenerator behaviour [2]:

$$\frac{\partial T(\xi,\eta)}{\partial \eta} = t(\xi,\eta) - T(\xi,\eta) \tag{1}$$

$$\frac{\partial t(\xi,\eta)}{\partial \xi} = T(\xi,\eta) - t(\xi,\eta). \tag{2}$$

In this solution, equation (2) is discretised using Euler's rule. An analytic approach is then used to solve the resultant set of ordinary differential equations in the independent variable η. This gives rise to a set of simultaneous, linear, algebraic equations. The number of linear equations is directly proportional to the number of steps required in the discretisation. Euler's rule is the least accurate of the finite-difference formulae [3] and, due to stability considerations, can only be implemented using a sufficiently small steplength. In order to achieve sufficient accuracy, and implicitly to avoid the effects of instability, Razelos [1] found it was necessary to use a steplength of the order 0.01 when representing equation (2); consequently the solution of up to 1000 linear equations was required.

This paper discusses a new method which uses the trapezoidal rule to discretise equation (2). The trapezoidal rule [3] is more accurate than Euler's and has no stability problems when used to replace equation (2). This approach not only substantially reduces the number of linear equations to be solved but also offers a solution to the long regenerator problem [4]. A significant advantage of this method for cyclic equilibrium calculations is that both the fluid and solid temperature distributions can be computed for any instant of time. This is not the case in the closed methods of Iliffe [5] and Nahavandi and Weinstein [6] where only the solid temperatures are available at the beginning and end of the hot and cold periods of operation.

2. THE MATHEMATICAL MODEL

The differential equations which model regenerator behaviour are:

$$hA(\phi - \Phi) = MC\frac{\partial \Phi}{\partial \theta} \tag{3}$$

$$hA(\Phi - \phi) = WSL\frac{\partial \phi}{\partial y} + mS\frac{\partial \phi}{\partial \theta}. \tag{4}$$

Built into equations (3) and (4) are the following idealisations and assumptions.

(i) The effects of the residual fluid in the matrix channels during the reversal periods are ignored.
(ii) The thermal conductivity of the fluids and matrix is zero in a direction parallel to that of the fluid stream.

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NOMENCLATURE

a	trapezoidal discretisation constant
A	heat transfer surface area [m^2]
A_n	constants of integration in general solution to equation (21)
b	trapezoidal discretisation constant
C	specific heat of matrix [$J\,kg^{-1}\,K^{-1}$]
h	heat transfer coefficient
K_n	general solution to equation (21)
L	length of regenerator [m]
m	mass of fluid in regenerator channels [kg]
M	mass of matrix [kg]
N	total number of regenerator segments
P	duration of a period [s]
R_n	particular solution to equation (21)
S	specific heat of fluid [$J\,kg^{-1}\,K^{-1}$]
t	dimensionless [0, 1] fluid temperature
T	dimensionless [0, 1] matrix temperature
W	mass flow rate of fluids [$kg\,s^{-1}$]
y	distance down the regenerator [m].

Greek symbols

α_j	series in η for general solution to equation (21)
β_j	differentiated α_j values
η	dimensionless time defined by equation (6)
η_{reg}	thermal ratio
ϕ	fluid temperature [K]
$\phi_0(\theta)$	inlet fluid temperature as a function of time [K]
ϕ_{max}	maximum value of $\phi_0(\theta)$ for $0 \leqslant \theta \leqslant P$ [K]
ϕ'_{min}	minimum value of $\phi'_0(\theta')$ for $0 \leqslant \theta' \leqslant P'$ [K]
θ	time [s]
ξ	dimensionless distance defined by equation (5)
Λ	reduced length defined by equation (7)
Π	reduced period defined by equation (8)
Φ	matrix temperature [K]
Ψ	transformation of T.

Superscript

′	refers to cold period.

Subscript

n	refers to nth node.

(iii) The solid temperature variation in the radial direction is not considered. It is assumed that the thermal conductivity in the radial direction is either infinite, in which case the solid will be isothermal in the radial direction, or finite. In the latter case a bulk heat transfer coefficient is used [7].
(iv) The heat transfer coefficient and thermal properties of both fluid and solid are regarded as temperature independent.
(v) The mass flow rate of the fluid in each period does not vary with time but may be different in the hot and cold periods.

By using the dimensionless parameter introduced by Hausen [2]:

$$\xi = \frac{hAy}{WSL} \tag{5}$$

$$\eta = \frac{hA}{MC}\left(\theta - \frac{my}{WL}\right) \tag{6}$$

and the dimensionless temperatures given by:

$$T = \frac{\Phi - \phi'_{min}}{\phi_{max} - \phi'_{min}}, \quad t = \frac{\phi - \phi'_{min}}{\phi_{max} - \phi'_{min}}$$

equations (3) and (4) take the form (1) and (2) given in Section 1. The values of ξ and η at $y = L$ and $\theta = P$ are:

$$\Lambda = \frac{hA}{WS} \tag{7}$$

$$\Pi = \frac{hA}{MC}\left(P - \frac{m}{w}\right) \tag{8}$$

which were named *reduced length* and *reduced period* respectively by Hausen [2]. The importance of the my/WL term in equation (6) is discussed in detail by Willmott and Hinchcliffe [8].

Any solution to equations (1) and (2) must also take into account the following boundary conditions:

(i) The inlet fluid temperature is predefined as some function of time, namely

$$t_0(\eta) = t(0, \eta). \tag{9}$$

(ii) Distances within a regenerator are measured from the fluid entrance in both periods. Further, the temperature distribution of the matrix at the end of one period is equal to the temperature distribution of the matrix at the beginning of the opposite period. For counter-current operation this results in the reversal conditions:

$$T(\xi, 0) = T'[\Lambda'(1-\xi/\Lambda), \Pi'] \tag{10}$$

$$T'(\xi', 0) = T[\Lambda(1-\xi'/\Lambda'), \Pi]. \tag{11}$$

Regenerator effectiveness

Regenerator effectiveness is measured in terms of the thermal ratio η_{reg} which describes the ratio of the actual heat transferred during a period to the thermodynamically limited maximum obtainable heat transfer for that period. This results in the two values of η_{reg} shown by Iliffe [5] to be:

$$\eta'_{reg} = \frac{\Lambda'}{\Pi'}\left[\frac{1}{\Lambda'}\int_0^{\Lambda'} T'(\xi',0)\,d\xi' - \frac{1}{\Lambda}\int_0^{\Lambda} T(\xi,0)\,d\xi\right] \tag{12}$$

$$\eta_{reg} = \frac{\Pi'\Lambda}{\Pi\Lambda'}\eta'_{reg}. \tag{13}$$

3. ANALYSIS

Discretising equation (2) employing the trapezoidal rule and considering the temperatures of fluid and solid at $(N+1)$ equidistant points or nodes along the regenerator yields:

$$\frac{dT_n}{d\eta} = t_n - T_n \quad (0 \leqslant n \leqslant N) \tag{14}$$

$$t_{n+1} = bt_n + a(T_n + T_{n+1}) \quad (0 \leqslant n \leqslant N-1) \tag{15}$$

with

$$a = \frac{\Delta\xi}{2+\Delta\xi}, \quad b = \frac{2-\Delta\xi}{2+\Delta\xi}, \quad (2a+b) = 1 \tag{16}$$

and

$$\Delta\xi = \frac{\Lambda}{N}, \quad \xi = n\Delta\xi. \tag{17}$$

The reversal conditions (10) and (11) become:

$$T_n(0) = T'_{N-n}(\Pi') \quad (0 \leqslant n \leqslant N) \tag{18}$$

$$T'_n(0) = T_{N-n}(\Pi) \quad (0 \leqslant n \leqslant N). \tag{19}$$

The transformation Ψ is now introduced and equation (14) is used to give:

$$\Psi_n = \exp(\eta)T_n \quad (0 \leqslant n \leqslant N) \tag{20}$$

$$\frac{d\Psi_n}{d\eta} = \exp(\eta)t_n \quad (0 \leqslant n \leqslant N) \tag{21}$$

which, using equation (15), becomes:

$$\frac{d\Psi_n}{d\eta} = a\Psi_n + a\Psi_{n-1} + b\frac{d\Psi_{n-1}}{d\eta} \quad (1 \leqslant n \leqslant N). \tag{22}$$

Any solution to equation (21) must satisfy this auxiliary equation (22).

Solution of equation (21)

The general solution to equation (21) (when $t_0 = 0$) is given by:

$$K_n = (-1)^n A_0 + \exp(a\eta)\left[A_n + \sum_{j=1}^{n-1} A_j\alpha_{n-j}\right] \quad (1 \leqslant n \leqslant N) \tag{23}$$

with

$$K_0 = A_0 \tag{24}$$

and

$$\alpha_r = b^r \sum_{k=1}^{r} \binom{r-1}{k-1}\left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!} \tag{25}$$

the A_j values being constants of integration.

When $t_0 = t_0(\eta)$ the particular solution to equation (21) is given by:

$$R_n = \exp(a\eta)\sum_{r=0}^{n-1} [a(1+b)]^{n-1-r} b^r \binom{n-1}{r} \times \int^{n-r} \exp(-a\eta)G_0(\eta)\,d\eta \quad (1 \leqslant n \leqslant N) \tag{26}$$

where

$$G_0(\eta) = a\int \exp(\eta)t_0(\eta)\,d\eta + b\exp(\eta)t_0(\eta) \tag{27}$$

$$R_0 = \int \exp(\eta)t_0(\eta)\,d\eta \tag{28}$$

and the symbol $\int^{n-r}$ indicates $(n-r)$ indefinite integrations of the function $\exp(-a\eta)G_0(\eta)$.

The proofs that equations (23) and (26) satisfy the auxiliary equation (22) are given in Appendix I. Given the general and particular solutions to equation (21), together with equation (20), all the required temperatures can be calculated.

Solution for constant inlet temperatures

In this case we have $t_0(\eta) = 1$ and $t'_0(\eta') = 0$ which yields:

$$G'_0(\eta') = 0, \quad G_0(\eta) = (a+b)\exp(\eta) = (1-a)\exp(\eta). \tag{29}$$

The particular solution R_n now takes the form:

$$R'_n = 0, \quad R_n = \exp(\eta). \tag{30}$$

From equations (20), (23) and (30) we obtain the solid temperatures:

$$T_n = \exp(-\eta)(-1)^n A_0 + \exp[(a-1)\eta]\left\{A_n + \sum_{j=1}^{n-1} A_j\alpha_{n-j}\right\} + 1 \tag{31}$$

$$T'_n = \exp(-\eta')(-1)^n A'_0 + \exp[(a'-1)\eta']\left\{A'_n + \sum_{j=1}^{n-1} A'_j\alpha'_{n-j}\right\} \tag{32}$$

and

$$T_0 = \exp(-\eta)A_0 + 1, \quad T'_0 = \exp(-\eta')A'_0. \tag{33}$$

For the gas temperatures equations (21), (23) and (30) are used to give:

$$t_n = a\exp[(a-1)\eta] \times \left[A_n + \sum_{j=1}^{n-1} A_j\{\alpha_{n-j} + (1+b)\beta_{n-j}\}\right] + 1 \tag{34}$$

$$t'_n = a'\exp[(a'-1)\eta'] \times \left[A'_n + \sum_{j=1}^{n-1} A'_j\{\alpha'_{n-j} + (1+b')\beta'_{n-j}\}\right] \tag{35}$$

where

$$\beta_r = b^{r-1}\sum_{k=0}^{r-1}\binom{r-1}{k}\left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!}. \tag{36}$$

Applying equations (31)–(33) to the reversal conditions (18) and (19) yields $2(N+1)$ equations in the $2(N+1)$ unknowns A_j, A'_j:

$$(-1)^n A_0 + A_n + 1 = \exp(-\Pi')(-1)^{N-n} A'_0 + \exp[(a'-1)\Pi']\left\{A'_{N-n} + \sum_{j=1}^{N-n-1} A'_j \alpha'_{N-n-j}\right\} \quad (37)$$

$$(-1)^n A'_0 + A'_n = 1 + \exp(-\Pi)(-1)^{N-n} A_0 + \exp[(a-1)\Pi]\left\{A_{N-n} + \sum_{j=1}^{N-n-1} A_j \alpha_{N-n-j}\right\}. \quad (38)$$

At the hot and cold fluid entrances, the corresponding equations are:

$$A_0 + 1 = \exp(-\Pi')(-1)^N A'_0 + \exp[(a'-1)\Pi']\left\{A'_N + \sum_{j=1}^{N-1} A'_j \alpha'_{N-j}\right\} \quad (39)$$

$$A'_0 = 1 + \exp(-\Pi)(-1)^N A_0 + \exp[(a-1)\Pi]\left\{A_N + \sum_{j=1}^{N-1} A_j \alpha_{N-j}\right\} \quad (40)$$

$$(-1)^N A_0 + A_N + 1 = \exp(-\Pi')A'_0 \quad (41)$$

$$(-1)^N A'_0 + A'_N = 1 + \exp(-\Pi)A_0. \quad (42)$$

In equations (37)–(40) the α_r, α'_r values are calculated at $\eta = \Pi$ and $\eta' = \Pi'$, respectively. In the important case of the symmetric regenerator, where $\Lambda = \Lambda'$ and $\Pi = \Pi'$, it can be shown that $(A_j + A'_j)$ is equal to zero and the number of equations required is halved. Further details concerning the implementation of equations (25), (31)–(42) are given in Appendix II.

Calculation of regenerator effectiveness η_{reg}

Equation (12) is evaluated by the trapezoidal rule using equation (43):

$$\eta'_{reg} = \frac{\Lambda'}{\Pi'}\frac{1}{N}\left\{\left(\frac{T'_0}{2} + \sum_{i=1}^{N-1} T'_i + \frac{T'_N}{2}\right) - \left(\frac{T_0}{2} + \sum_{i=1}^{N-1} T_i + \frac{T_N}{2}\right)\right\}. \quad (43)$$

In the case of constant inlet temperatures η'_{reg} can also be calculated from the cold exit fluid temperatures using equation (44):

$$\eta'_{reg} = \frac{1}{\Pi'}\int_0^{\Pi'} t'_N(\eta')\,d\eta'. \quad (44)$$

Equation (43) is generally used in preference to equation (44) as the temperatures T_j, T'_j at $\eta = 0$ can be calculated from the A_j, A'_j values directly, the α_r, α'_r values being zero. However, when both the Λ/Π and Λ'/Π' ratios become very large a loss of precision is associated with equation (43). In this case the exit fluid temperature variation is linear and equation (44) should be used.

4. RESULTS

The results for symmetric, counter-flow regenerators with constant inlet temperatures reported by Razelos [1] have been computed. Similar results were obtained employing much smaller values of N, typically 10–50 compared with values of 400–1000 used in the former work. As an example the symmetric case $\Lambda = 10, \Pi = \pi$ is presented in Table 1. In this particular case the value of N used was 30.

Many non-symmetric, counter-flow cases have also been computed with a comparable economy in N. The cases presented by Nahavandi and Weinstein [6] have been computed and the results are given in Table 2. This table was produced using values of N between 10 and 40 to obtain agreement in η_{reg} values of 0.0001. The results can be seen to be in good agreement with previous work [9].

In order to gauge fully the computational performance of the present method, a comparison has been made with the open scheme of Willmott [10]. In open methods the periodic behaviour of the regenerator is emulated directly, the model being cycled to equilibrium in the same way that the regenerator cycles to equilibrium. In the Willmott method, both equations (1) and (2) are integrated using the trapezoidal rule and thus the error associated with the discretisation of equation (2) is the same as in the method discussed here. The open method was implemented, using the convergence criteria described by Willmott and Burns [11], by fixing the number of distance steps (i.e. N) and increasing the number of time steps by 10 until successive η_{reg} values differed by less than 0.0001. In this way the time taken to compute a converged η_{reg} value for a given N could be ascertained and compared with that required by the closed method for the same value of N.

In Table 3 the relative CPU times taken by the two methods to compute the following table of η_{reg} values, using different values of N and k, are presented.

Table 1. Matrix temperatures during the cold period for a symmetric, counter-flow regenerator with $\Lambda = 10$, $\Pi = \pi$, $N = 30$

	η'				
ξ'	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
0	0.2222	0.1013	0.0462	0.0211	0.0096
1	0.2977	0.2060	0.1365	0.0877	0.0550
2	0.3793	0.3011	0.2287	0.1675	0.1192
3	0.4644	0.3919	0.3199	0.2532	0.1949
4	0.5509	0.4807	0.4098	0.3410	0.2771
5	0.6375	0.5685	0.4986	0.4293	0.3625
6	0.7229	0.6549	0.5861	0.5171	0.4491
7	0.8051	0.7389	0.6716	0.6036	0.5356
8	0.8808	0.8182	0.7536	0.6875	0.6207
9	0.9450	0.8889	0.8292	0.7667	0.7023
10	0.9904	0.9452	0.8938	0.8376	0.7778

Table 2. Values of η'_{reg} for counter-flow regenerators. The number of nodes is increased by 10 until successive estimates of η'_{reg} differ by less than 0.0001

Λ	Λ'	Π	Π'	Number of nodes 10	20	30	40	Lambertson [9]
1.4	2	2	2	0.4960	0.4960			0.4959
2.8	4	4	4	0.6437	0.6437			0.6437
5.6	8	8	8	0.7588	0.7589	0.7590		0.7590
11.2	16	16	16	0.8400	0.8404	0.8404		0.8405
1.4	2	1	1	0.5271	0.5271			0.5271
2.8	4	2	2	0.7088	0.7087			0.7087
5.6	8	4	4	0.8538	0.8536	0.8536		0.8535
11.2	16	8	8	0.9463	0.9458	0.9457	0.9456	0.9456
0.2	2	2	2	0.5536	0.5535			0.5534
0.4	4	4	4	0.7072	0.7070	0.7070		0.7070
0.8	8	8	8	0.8014	0.8011	0.8010		0.8009
1.6	16	16	16	0.8601	0.8597	0.8596		0.8595
0.2	2	1	1	0.6012	0.6010	0.6010		0.6008
0.4	4	2	2	0.8117	0.8112	0.8111		0.8109
0.8	8	4	4	0.9412	0.9401	0.9399	0.9398	0.9397
1.6	16	8	8	0.9897	0.9880	0.9877	0.9876	0.9874
1.0	2	2	2	0.5156	0.5156			0.5156
2.0	4	4	4	0.6690	0.6689			0.6690
4.0	8	8	8	0.7817	0.7816			0.7817
8.0	16	16	16	0.8548	0.8547	0.8546		0.8546
1.0	2	1	1	0.5516	0.5516			0.5515
2.0	4	2	2	0.7463	0.7462	0.7462		0.7461
4.0	8	4	4	0.8935	0.8930	0.8929		0.8928
8.0	16	8	8	0.9731	0.9720	0.9718	0.9717	0.9717

$$\Lambda = 1, 2, \ldots, 10; \quad \Pi = 1, 2, 3;$$
$$\Lambda' = k \cdot \Lambda; \qquad \Pi' = k \cdot \Pi.$$

It can be seen from Table 3 that the time taken by the closed method compares favourably with that required by the open scheme, especially in the case of the symmetric regenerator ($k = 1$). The improved performance of the present method in the case of the symmetric regenerator is to be expected as the solution of only $(N+1)$ equations is required. The relative decrease in performance of the closed scheme as N increases is attributable to the difference in the order of the workload in both schemes. In general the workload of the present method is proportional to N^3 and in the Willmott scheme is approximately cNP, where P is the number of steps required in the discretisation of equation (1) and c is the number of cycles required to reach equilibrium. Hence, as N becomes larger the computational time required by the present method increases more rapidly than in the Willmott scheme. As values of N between 10 and 30 cover most practical applications of the Willmott method, Table 3 suggests that the closed scheme discussed here is, in general, better suited to cyclic equilibrium calculations than the open Willmott method.

As the Λ/Π ratio becomes larger the open scheme takes longer to cycle to equilibrium and the usefulness of the closed method becomes more apparent. Table 4 is presented in order to show the superior ability of the closed scheme under these circumstances. The parameters used in Table 4 are:

$$\Lambda = 10, 20, \ldots, 50; \quad \Pi = 1;$$
$$\Lambda' = k \cdot \Lambda; \qquad \Pi' = k \cdot \Pi.$$

As mentioned previously, the present method also offers a solution to the problem of the long

Table 3. Relative CPU times, for the open Willmott scheme and the present method, taken to compute the table of η_{reg} values $\Lambda = 1, 2, \ldots, 10$; $\Pi = 1, 2, 3$

$\frac{\Lambda'}{\Lambda} = \frac{\Pi'}{\Pi} = k$	Hill-Willmott (1986) / Willmott [10] $N = 10$	$N = 20$	$N = 30$
1	0.13	0.17	0.24
1.5	0.43	0.79	1.23
2	0.43	0.78	1.22

Table 4. Relative CPU times, for the open Willmott scheme and the present method, taken to compute the table of η_{reg} values $\Lambda = 10, 20, \ldots, 50$; $\Pi = 1$

$\frac{\Lambda'}{\Lambda} = \frac{\Pi'}{\Pi} = k$	Hill-Willmott (1986) / Willmott [10] $N = 10$	$N = 20$	$N = 30$
1	0.06	0.06	0.09
1.5	0.12	0.21	0.35
2	0.11	0.19	0.31

Table 5. Matrix temperatures at $\eta = 0$ and $\eta = \Pi$ during the hot period for symmetric, counter-flow regenerators with $\Pi = 0.1$, $N = 40$

	$\eta = 0$			$\eta = \Pi$		
ξ/Λ	$\Lambda = 100$	$\Lambda = 250$	$\Lambda = 500$	$\Lambda = 100$	$\Lambda = 250$	$\Lambda = 500$
0.0	0.9897	0.9958	0.9979	0.9907	0.9962	0.9981
0.1	0.8917	0.8966	0.8982	0.8927	0.8970	0.8984
0.2	0.7936	0.7974	0.7986	0.7946	0.7978	0.7988
0.3	0.6956	0.6982	0.6990	0.6966	0.6986	0.6992
0.4	0.5976	0.5990	0.5995	0.5985	0.5994	0.5997
0.5	0.4995	0.4998	0.4999	0.5005	0.5002	0.5001
0.6	0.4015	0.4006	0.4003	0.4024	0.4010	0.4005
0.7	0.3034	0.3014	0.3008	0.3044	0.3018	0.3010
0.8	0.2054	0.2022	0.2012	0.2064	0.2026	0.2014
0.9	0.1073	0.1030	0.1016	0.1083	0.1034	0.1018
1.0	0.0093	0.0038	0.0019	0.0103	0.0042	0.0021

Table 6. Fluid temperatures at $\eta = 0$ and $\eta = \Pi$ during the hot period for symmetric, counter-flow regenerators with $\Pi = 0.1$, $N = 40$

	$\eta = 0$			$\eta = \Pi$		
ξ/Λ	$\Lambda = 100$	$\Lambda = 250$	$\Lambda = 500$	$\Lambda = 100$	$\Lambda = 250$	$\Lambda = 500$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.9015	0.9006	0.9003	0.9025	0.9010	0.9004
0.2	0.8034	0.8014	0.8006	0.8044	0.8018	0.8008
0.3	0.7054	0.7022	0.7010	0.7064	0.7026	0.7012
0.4	0.6074	0.6030	0.6014	0.6083	0.6034	0.6016
0.5	0.5093	0.5038	0.5019	0.5103	0.5042	0.5021
0.6	0.4113	0.4046	0.4023	0.4123	0.4050	0.4025
0.7	0.3132	0.3054	0.3028	0.3142	0.3058	0.3030
0.8	0.2152	0.2062	0.2032	0.2162	0.2066	0.2034
0.9	0.1171	0.1070	0.1036	0.1181	0.1074	0.1038
1.0	0.0191	0.0077	0.0039	0.0201	0.0081	0.0041

regenerator. Willmott and Thomas [4] define a long regenerator as having a reduced length Λ greater than 10 with the Λ/Π ratio greater than 3. For very long regenerators the closed methods of Iliffe [5] and Nahavandi and Weinstein [6] break down and a scheme such as that proposed by Hausen [12] can be adopted. The Hausen method, as discussed by Willmott and Thomas [4], assumes that the spatial and chronological variations of the solid temperature in the middle of a long regenerator are strictly linear. Hausen computes the performance of an equivalent shorter regenerator and then extrapolates the results calculated to the long regenerator case by exploiting this assumption. More recently Baclic [13] describes a closed solution which applies the Galerkin method to the Nusselt [14] equations. In the results reported by Baclic, however, the largest Λ/Π ratio considered is 10, whereas the method presented in this paper can deal with Λ/Π ratios of the order 5000.

Examples of the ability of the present method to compute correctly the solution to this problem are presented in Tables 5–7. They show the temperature distributions of the solid (Table 5) and fluid (Table 6) for values of reduced length up to 500 with a reduced period of 0.1. The values of thermal ratio η_{reg} are calculated using equation (44) and are presented in Table 7. It can be seen that the computed η_{reg} values agree, as would be expected, very closely with the ideal thermal ratio (when $\Pi = 0$) for symmetric, counter-flow regenerators which is given by $\Lambda/(\Lambda+2)$. Even in these cases, a value of $N = 40$ is sufficient to generate accurate solutions.

5. CONCLUSIONS

The use of the trapezoidal rule to represent equation (2) yields a robust, stable solution to the regenerator

Table 7. Exit fluid temperatures during the cold period for symmetric, counter-flow regenerators using $N = 40$

	Exit fluid temperatures		
η	$\Lambda = 100$	$\Lambda = 250$	$\Lambda = 500$
0.000	0.98088	0.99226	0.99611
0.025	0.98064	0.99216	0.99606
0.050	0.98039	0.99206	0.99601
0.075	0.98015	0.99196	0.99596
0.100	0.97990	0.99186	0.99591
Average	0.98039	0.99206	0.99601
$\Lambda/(\Lambda+2)$	0.98039	0.99206	0.99602

model. From the results it can be seen that the present method allows a considerable reduction in the amount of computational effort required to calculate accurate solutions to the model, when compared with that proposed by Razelos [1]. Not only is this significant in itself but the range of the dimensionless parameters Λ and Π for which the method is applicable is very large and includes cases which were previously beyond computation.

Although for long regenerators, with very small reduced periods, it is possible to estimate accurately the thermal ratio η_{reg} from the ideal thermal ratio given above, this ideal is not applicable to larger reduced periods. More important, in all cases, it is not possible to estimate the spatial and chronological variation of solid and fluid temperatures at cyclic equilibrium without solving equations (1) and (2). This new closed method enables the temperatures to be computed directly for cyclic equilibrium without running the model through many previous periods of operation.

Extensions

For parallel-flow regenerators the reversal conditions are given by:

$$T_n(0) = T_n'(\Pi') \quad (0 \leqslant n \leqslant N)$$

$$T_n'(0) = T_n(\Pi) \quad (0 \leqslant n \leqslant N).$$

The application of equations (31)–(33) to the above reversal conditions results in $2(N+1)$ equations in the $2(N+1)$ unknowns A_j, A_j' from which the A_j, A_j' values can be computed directly. The implementation of this scheme has been found to yield a rapid method for parallel-flow regenerator calculations.

One of the important cases considered by Razelos [1] is the expression of the inlet fluid temperatures as a polynomial of time, i.e.

$$t_0(\eta) = \sum_{i=0}^{s} c_i \eta^i = Z_0(\eta).$$

Making the substitution $\tau = (1-a)\eta$ in equation (26) gives:

$$R_n = \exp(a\eta) \sum_{r=0}^{n-1} (1-b)^{n-r-1} b^r \binom{n-1}{r} \times \int^{n-r} \exp(\tau) Z_2(\tau)\,\mathrm{d}\tau$$

where

$$Z_2(\tau) = \frac{Z_1[\tau/(1-a)]}{(1-a)}$$

and

$$Z_1(\eta) = a \sum_{i=0}^{s} (-1)^i \frac{\mathrm{d}^i Z_0(\eta)}{\mathrm{d}\eta^i} + b Z_0(\eta).$$

The form of the particular solution in this case is very similar to that given by Razelos [1].

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APPENDIX I

(i) Proof that equation (23) satisfies the auxiliary equation (22):

$$K_n = (-1)^n A_0 + \exp(a\eta)\left\{A_n + \sum_{j=1}^{n-1} A_j b^{n-j} \times \sum_{k=1}^{n-j} \binom{n-j-1}{k-1} \left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!}\right\}$$

$$\frac{\mathrm{d}K_n}{\mathrm{d}\eta} = a[K_n - (-1)^n A_0] + a(1+b) \times \exp(a\eta)\left\{\sum_{j=1}^{n-1} A_j b^{n-j-1} \sum_{k=1}^{n-j} \binom{n-j-1}{k-1} \times \left(\frac{a(1+b)\eta}{b}\right)^{k-1} \frac{1}{(k-1)!}\right\}.$$

The A_{n-1} term is now extracted from the series:

$$\frac{dK_n}{d\eta} = a[K_n-(-1)^nA_0]+a(1+b)\times\exp(a\eta)\left\{A_{n-1}+\sum_{j=1}^{n-2}A_jb^{n-j-1}\sum_{k=1}^{n-j}\binom{n-j-1}{k-1}\times\left(\frac{a(1+b)\eta}{b}\right)^{k-1}\frac{1}{(k-1)!}\right\}.$$

Using the binomial relationship

$$\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$$

we obtain:

$$\frac{dK_n}{d\eta} = a[K_n-(-1)^nA_0]+a(1+b)\times\exp(a\eta)\Bigg\{A_{n-1}+\sum_{j=1}^{n-2}A_jb^{n-j-1}\sum_{k=2}^{n-j}\binom{n-j-2}{k-2}\times\left(\frac{a(1+b)\eta}{b}\right)^{k-1}\frac{1}{(k-1)!}+\sum_{j=1}^{n-2}A_jb^{n-j-1}\sum_{k=1}^{n-j-1}\binom{n-j-2}{k-1}\times\left(\frac{a(1+b)\eta}{b}\right)^{k-1}\frac{1}{(k-1)!}\Bigg\}.$$

By using the substitution $r = k-1$ in the first series it can be seen that:

$$\frac{dK_n}{d\eta} = a[K_n-(-1)^nA_0]+a(1+b)\Bigg\{(K_{n-1}-(-1)^{n-1}A_0)+\exp(a\eta)\sum_{j=1}^{n-2}A_jb^{n-j-1}\times\sum_{k=1}^{n-j-1}\binom{n-j-2}{k-1}\left(\frac{a(1+b)\eta}{b}\right)^{k-1}\frac{1}{(k-1)!}\Bigg\}$$

$$\frac{dK_n}{d\eta} = aK_n+aK_{n-1}+b\Bigg\{a[K_{n-1}-(-1)^{n-1}A_0]+a(1+b)\exp(a\eta)\sum_{j=1}^{n-2}A_jb^{n-j-2}\times\sum_{k=1}^{n-j-1}\binom{n-j-2}{k-1}\left(\frac{a(1+b)\eta}{b}\right)^{k-1}\frac{1}{(k-1)!}\Bigg\}$$

$$\frac{dK_n}{d\eta} = aK_n+aK_{n-1}+b\frac{dK_{n-1}}{d\eta}.$$ Q.E.D.

(ii) Proof that equation (26) satisfies the auxiliary equation (22):

$$R_n = \exp(a\eta)\sum_{r=0}^{n-1}[a(1+b)]^{n-1-r}b^r\binom{n-1}{r}\times\int^{n-r}\exp(-a\eta)G_0(\eta)\,d\eta$$

$$\frac{dR_n}{d\eta} = aR_n+\exp(a\eta)\sum_{r=0}^{n-1}[a(1+b)]^{n-1-r}b^r\binom{n-1}{r}\times\int^{n-1-r}\exp(-a\eta)G_0(\eta)\,d\eta.$$

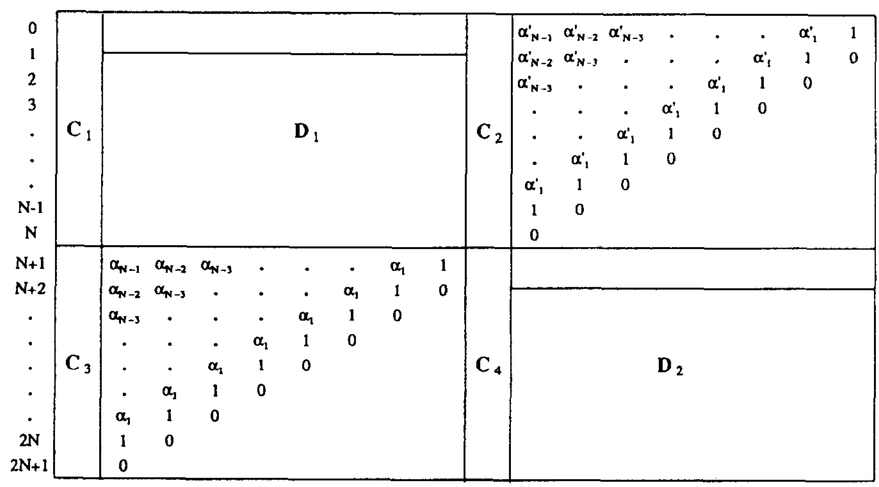

FIG. A1. Matrix **M**: blank areas in the matrix represent zero values. Note: for symmetric regenerators $A_j + A'_j = 0$ and only the top or bottom 'half' of the matrix need be considered.

Key

$\mathbf{C}_1$ $[-(-1)^n\exp((1-a')\Pi')]^T$	$(0 \leqslant n \leqslant N)$
$\mathbf{C}_2$ $[(-1)^{N-n}\exp(-a'\Pi')]^T$	$(0 \leqslant n \leqslant N)$
$\mathbf{C}_3$ $[(-1)^{N-n}\exp(-a\Pi)]^T$	$(0 \leqslant n \leqslant N)$
$\mathbf{C}_4$ $[-(-1)^n\exp((1-a)\Pi)]^T$	$(0 \leqslant n \leqslant N)$
$\mathbf{D}_1$ Diagonal matrix, diagonal elements	$-\exp[(1-a')\Pi']$
$\mathbf{D}_2$ Diagonal matrix, diagonal elements	$-\exp[(1-a)\Pi]$

Using the binomial relationship

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

we obtain:

$$\frac{dR_n}{d\eta} = aR_n + \exp(a\eta)\left\{\sum_{r=0}^{n-2}[a(1+b)]^{n-1-r}b^r\binom{n-2}{r} \times \int^{n-1-r}\exp(-a\eta)G_0(\eta)\,d\eta + \sum_{r=1}^{n-1}[a(1+b)]^{n-1-r}b^r\binom{n-2}{r-1} \times \int^{n-1-r}\exp(-a\eta)G_0(\eta)\,d\eta\right\}.$$

Using the substitution $k = r-1$ in the second series we find:

$$\frac{dR_n}{d\eta} = aR_n + \exp(a\eta)\left\{a(1+b)\sum_{r=0}^{n-2}[a(1+b)]^{n-2-r}b^r\binom{n-2}{r} \times \int^{n-1-r}\exp(-a\eta)G_0(\eta)\,d\eta + b\sum_{k=0}^{n-2}[a(1+b)]^{n-2-k}b^k\binom{n-2}{k} \times \int^{n-2-k}\exp(-a\eta)G_0(\eta)\,d\eta\right\}$$

$$\frac{dR_n}{d\eta} = aR_n + aR_{n-1} + b\left\{aR_{n-1} + \exp(a\eta)\sum_{k=0}^{n-2}[a(1+b)]^{n-2-k}b^k\binom{n-2}{k} \times \int^{n-2-k}\exp(-a\eta)G_0(\eta)\,d\eta\right\}$$

$$\frac{dR_n}{d\eta} = aR_n + aR_{n-1} + b\frac{dR_{n-1}}{d\eta}. \qquad \text{Q.E.D.}$$

APPENDIX II

Equations (25), (31)–(42) are implemented in the following manner:

(a) The α_r, α_r' values are calculated from equation (25) using $(a, b, \eta = \Pi)$ and $(a', b', \eta' = \Pi')$, respectively.

(b) The system of equations $\mathbf{MA} = \mathbf{y}$ is then set up using equations (37)–(42). The form of $\mathbf{M}$ is given in Fig. A1 and $\mathbf{y}$ is defined below. $\mathbf{A}$ is the vector of unknowns $[A_0 \cdots A_N,\ A_0' \cdots A_N']^{\mathrm{T}}$. The set of equations can be solved using a library routine or the Gauss–Seidel iterative scheme—depending on the Λ/Π ratio. In this paper the iterative scheme was used for $\{\Lambda/\Pi \leqslant 5$ and $\Pi \geqslant 1\}$ and the Crout factorisation library routine otherwise.

Define

$$\mathbf{e}_1^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 2 & \ldots & N & N+1 & N+2 & N+3 & \ldots & 2N+1 \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$\mathbf{e}_2^{\mathrm{T}} = [0 \ \ 0 \ \ 0 \ \ldots \ 0 \ \ 1 \ \ 1 \ \ 1 \ \ldots \ 1]$$

Then

$$\mathbf{y} = \exp[(1-a')\Pi']\mathbf{e}_1 - \exp[(1-a)\Pi]\mathbf{e}_2.$$

(c) The temperatures at any time η can now be computed by first calculating the α_r, β_r values from equations (25) and (36) and then using equations (31)–(33) (solid) and equations (34) and (35) (fluid).

UNE METHODE ROBUSTE POUR LE CALCUL D'UN ECHANGEUR-REGENERATEUR

Résumé—On discute une méthode stable et précise pour la résolution des équations différentielles décrivant le comportement d'un régénérateur de chaleur. L'approche adoptée est reliée à celle proposée par Razelos, mais elle emploie la règle trapézoïdale pour fournir la précision et, en particulier, la stabilité numérique. Des détails de la nouvelle méthode sont décrits avec l'application à un large domaine des paramètres descriptifs.

EIN ROBUSTES VERFAHREN ZUR BERECHNUNG VON REGENERATIVEN WÄRMEAUSTAUSCHERN

Zusammenfassung—Eine stabile und genaue, geschlossene Methode zur Lösung der Differentialgleichungen, welche das Verhalten eines thermischen Regenerators beschreiben, wird vorgestellt. Das angewandte Verfahren lehnt sich an die Methode von Razelos [*Wärme- u. Stoffübertr.* **12**, 59–71 (1979)] an; hier wird jedoch die Trapezregel verwendet, um mehr Genauigkeit und insbesondere mehr numerische Stabilität zu gewinnen. Die Einzelheiten des neuen Verfahrens werden beschrieben, außerdem dessen Anwendung in einem weiten Parameterbereich.

ЭФФЕКТИВНЫЙ МЕТОД РАСЧЕТА РЕГЕНЕРАТИВНЫХ ТЕПЛООБМЕННИКОВ

Аннотация—Предложен метод получения устойчивых с заданной точностью решений дифференциальных уравнений, описывающих режим работы регенеративного теплообменника. Методика решения аналогична предложенной Рацелозом [*Wärme- u. Stoffübertr.* **12**, 59–71 (1979)], но используется еще формула трапеций для обеспечения точности и, в частности, устойчивости численного решения. Дано подробное описание метода, а также показано его применение в широком диапазоне значений основных параметров.